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Theory of quantum phenomena via extended measures: geometric features

S K Srinivasan

Department of Mathematics, Indian Institute of Technology, Madras, Chennai 600 036, India

E-mail: rangan@iitm.ac.in

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Abstract

The formulation of quantum phenomena via extended measures is reviewed and the motion of an electron in a uniform magnetic field is modelled in the quaternion measure theoretic framework. It is shown that there are some geometric transformations that enable us to conjecture that forces of electromagnetic origin may be a manifestation of the geometrical property of the measure structure.

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1. Introduction

The aim of this paper is to establish the viability of describing the various facets of quantum phenomena through extended measures. In earlier papers (Srinivasan and Sudarshan 1994, 1996, Srinivasan 1997) extended measures, and measurable processes arising therefrom, were studied with the specific aim of exploring their appropriateness in describing quantum phenomena; early signals from such a study point to an interesting scenario; the very complex nature of the basic measure employed is able to bring out interference and accommodate violation of Bell's inequalities (Youssef 1994, 1995) in sharp contrast with the traditional positive definite measurable processes. It turns out (Srinivasan 1995, 1997) that the measure density corresponding to a forced harmonic oscillator admits an explicit expression with a characteristic asymptotic behaviour that eliminates the need for the ultraviolet cut-off and provides a convergent result for the Lamb-shift without recourse to any renormalization procedure. Apart from this, a model of a hydrogen atom and the motion of a charged particle subject to a constant magnetic field have also been studied in the framework of complex measure theory (Srinivasan 2001). In this paper, we wish to focus our attention on the general analysis of the basic equations satisfied by the measure density. Specifically we demonstrate that forces of electromagnetic origin may arise by a transformation of the coordinates employed to characterize the measure density and that consequently a particle subject to a force may be visualized as a free particle in a suitable product frame of reference. Thus we conjecture that the very measure structure employed to describe a phenomenon is viable enough to encompass in itself appropriate geometric transformation properties consistent with the principle of general relativity.

The motivation for the present approach stems from the ground reality that quantum phenomena, as emerged through the early 20th century, are more an indication that probability theory is not the right tool to model microphysics rather than the weighty historical evolution by invoking quantum mechanics. The basic idea that probability theory needs modification in order to make it an applicable tool to explain quantum phenomena is not entirely new and perhaps goes back to Dirac (1942) (see also Youssef (2001)). More recently, Youssef (1991, 1994, 1995) has made out a case for the description of quantum phenomena through the use of 'exotic' probability theory. Our formulation of quantum phenomena through complex measures and complex measurable processes in the first instance (Srinivasan and Sudarshan 1994, Srinivasan 1997) is simple in that it is the minimal extension of positive definite measure that accommodates interference and ensures violation of Bell's inequalities. However, it does not bring out the characteristics of internal motion such as intrinsic spin. To accommodate the same, a further extension of the measure structure is necessary and this is best done by making a further generalization to quaternion measure and measurable processes. Although the quaternion measure can be thought of as a vector measure, such measurable processes have not been studied in the past. In earlier contributions (Srinivasan and Sudarshan 1996, Srinivasan 1995, 1997) it has been shown that Pauli and Dirac structures can be brought out by simple quaternion measurable processes that have the Markov property. In fact, quaternionic analysis has been used in the literature (see De Leo and Rodrigues (1998a)) to bring out the geometrical properties of the Dirac equation. However, our approach is distinct in that it starts from a basic quaternion measure structure and identifies the standard equations as special cases of the Fokker–Planck (FP) equation. A detailed account of quaternion measurable processes can be found in Srinivasan (1995) and Srinivasan and Sudarshan (1996). To facilitate a proper understanding, in the next section we provide a short summary of complex and quaternion measurable processes that leads to the standard equations.

The layout of the paper is as follows. In section 2, a quick introduction to extended measure theoretic framework is provided leading to the quaternion measure theoretic derivation of the Dirac equation. The motion of a quantum oscillator is discussed with reference to the transformation properties. In section 3, the motion of an electron in a uniform magnetic field is described in the quantum measure theoretic framework (QMTF). The non-relativistic approximation is shown to be consistent with the conventional Schrödinger–Pauli approach. An explicit solution for the resulting equation is obtained in a closed form. The transformation that leads to the solution is interpreted as the one that also leads to the characterization of a free particle. The full relativistic version is also shown to lead to a free particle equation under appropriate transformation of the product space of space–time and quaternion space over which the basic measure density is defined.

2. Complex/quaternion measure

We first summarize the broad features of complex measure theoretic framework (CMTF). We start with a measurable space $(\Omega, \boldsymbol{B})$; if μ_1 and μ_2 are any two signed measures defined over $(\Omega, \boldsymbol{B})$, then the complex measure λ is defined by $\lambda = \mu_1 + \mathrm{i}\mu_2$. Random variables and stochastic processes are introduced in exactly the same manner as in the case of non-negative measure (see, for example, Pitt (1963)); more details are provided in Srinivasan and Sudarshan (1994). Next it is to be noted that the constraint $\lambda(\Omega) = 1$ is imposed on the measure structure; this ensures that the Chapman–Kolmogorov relation can be translated into a good differential equation. Apart from this, a further constraint $|\lambda(A)| < \infty$ is imposed; this is just a formality and there are many exceptions, such as the free particle case.

To get to the central point, we note that we generally deal with the complex measure density (CMD) of the coordinate. The dynamics is brought out by assuming that the stochastic process of the coordinate is Markov in nature and essentially determined by the drift and diffusion functions denoted respectively by $A(x)$ and $B(x)$. This in turn yields a differential equation of the FP type for the CMD function governing the coordinate. The only cases that have been covered so far are:

(i) free harmonic oscillator $\{A(x) = -\mathrm{i}\omega x,\ B(x) = \text{a constant} = \mathrm{i}\hbar/m\}$;
(ii) displaced/squeezed oscillator $\{A(x) = -\mathrm{i}\omega(x-\alpha),\ B(x) = \text{a constant}\}$;
(iii) forced oscillator $\{A(x) = -\mathrm{i}\omega x + F(t),\ B(x) = \text{a constant}\}$.

Encouraged by this, we have identified the quantum harmonic oscillator as a complex measured diffusion process; interestingly the CMD becomes connected to the Feynman path integral solution (Feynman and Hibbs 1965). In view of this it was thought worthwhile to investigate the forced harmonic oscillator since the forcing term can be chosen as the matter current in interaction with the field of quantum oscillators. Using strong constraints on the CMD which are eventually satisfied in the case of the forced oscillator, the Hilbert space approach is employed. It turns out (Srinivasan 1997) that the results can be put in correspondence with those of Feynman and Hibbs (1965) particularly in the context of Lamb-shift calculations. However, there is one significant difference; the integral over momentum converges, eliminating the need for any ultraviolet cut-off. This provides an effective alternate way out since the calculations from that level can be taken over *in toto*, the celebrated Feynman cut-off procedure itself being capable of interpretation as a valid numerical approximation on the understanding that CMTF is an independent basis for the interpretation of quantum phenomena. Apart from these, the hydrogen atom and the motion of a charged particle subject to a constant magnetic field have also been investigated within the framework of complex measurable processes.

To accommodate the degree of freedom due to intrinsic spin, the complex measure is replaced by the quaternion measure where the quaternions are defined over the complex field. As observed earlier the primary motivation for the introduction of the quaternion valued measure stems from the fact that a particle with spin has intrinsic geometry as evidenced by the rotational characteristics of the extra degrees of freedom. That a spinning particle such as an electron has a characteristic jittery motion (zitterbewegung) is well recognized by physicists starting with Dirac and serves as the guiding physical principle for modelling the free spinning motion as a two-valued quaternion measurable process. At the outset it is worth mentioning that this development describing zitterbewegung in terms of a two-valued quaternion measure process does not have any connection to the quaternionic theory of electrons due to Hestenes (1966), Adler (1995) and De Leo and Rodrigues (1998b) who essentially interpret Dirac structure in terms of the geometry of the quaternions. Let us now briefly review our earlier work on quaternionic measure.

In quaternion measure theoretic framework, we visualize the stochastic process as the fusion of a two-valued process (corresponding to the helicity states) and a continuous drift process describing the intrinsic dispersion. Thus instead of the CMD we start with the quaternion measure density $\pi_+(\boldsymbol{x}, t)(\pi_-(\boldsymbol{x}, t))$, where $\pi_+(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}(\pi_-(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x})$ represents the quaternion measure that the particle is situated in $(\boldsymbol{x}, \boldsymbol{x}+\mathrm{d}\boldsymbol{x})$ and that the two-valued process $Z(t)$ takes the value 0(1) at position $\boldsymbol{x}$. We assume that the process $Z(t)$ is independent of the process $\boldsymbol{x}$ and the drift is given in terms of conditional structure:

$$\mathrm{E}[x_j|Z(t) = 0] = \mathrm{E}[V_j(\boldsymbol{x})]\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t) \tag{1}$$

$$\mathrm{E}[x_j|Z(t) = 1] = -\mathrm{E}[V_j(\boldsymbol{x})]\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t). \tag{2}$$

Next we choose

$$\mathrm{E}[V_j(\boldsymbol{x})] = c\sigma_j. \tag{3}$$

By adapting the FP method, we obtain

$$\partial_t \pi_+(\boldsymbol{x}, t) = -c\vec{\boldsymbol{\sigma}} \cdot \nabla(\pi_+(\boldsymbol{x}, t) - \{\lambda_+\pi_+(\boldsymbol{x}, t) - \lambda_-\pi_-(\boldsymbol{x}, t)\}, \tag{4}$$

$$\partial_t \pi_-(\boldsymbol{x}, t) = c\vec{\boldsymbol{\sigma}} \cdot \nabla(\pi_-(\boldsymbol{x}, t) - \{\lambda_-\pi_-(\boldsymbol{x}, t) - \lambda_+\pi_+(\boldsymbol{x}, t)\}. \tag{5}$$

At the outset we note that the π-functions are quaternion valued and can be represented by the σ-matrices; we can post-multiply by arbitrary spinors to yield two-component objects (two-spinors). Thus if we set

$$\lambda_+ = \lambda_- = -\mathrm{i}mc^2/\hbar \tag{6}$$

$$\psi = \begin{bmatrix} \pi_+ \\ \pi_- \end{bmatrix} \exp\{-\mathrm{i}mc^2 t/\hbar\} \tag{7}$$

we obtain the Dirac equation in four-component form in the Weyl representation:

$$\mathrm{i}\hbar\partial_t\psi = mc^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi + (\hbar c/\mathrm{i}) \begin{bmatrix} \vec{\boldsymbol{\sigma}} & 0 \\ 0 & -\vec{\boldsymbol{\sigma}} \end{bmatrix} \cdot \nabla\psi. \tag{8}$$

It is worth mentioning that Weyl representation is generally not in common use. Pauli (1980) in his exposition of quantum mechanics used the Weyl representation at first. However, he found that while the structure had good transformation properties under proper Lorentz transformation, in the sense that the four-component system behaves as two units of two vectors, under parity transformation the units became interchanged. This situation forced him to revert back to the original Dirac representation. However, in QMTF, the two blocks correspond to the two helicity states and admit a direct interpretation from a generalized probabilistic point of view.

We now deal with the central theme; we first consider for the sake of clarity the free harmonic oscillator in one dimension. If $\pi(x, t)$ is the CMD of the coordinate x at time t, then by the assumptions stated above we have the FP equation (Srinivasan 1997):

$$\frac{\partial\pi(x, t)}{\partial t} = \frac{\partial}{\partial x}[\mathrm{i}\omega x\pi(x, t)] + \frac{\mathrm{i}\hbar}{2m}\frac{\partial^2\pi(x, t)}{\partial x^2}. \tag{9}$$

Equation (9) is solved by the transformation

$$\xi = x\mathrm{e}^{\mathrm{i}\omega t}, \qquad \tau = \mathrm{e}^{\mathrm{i}\omega t}, \qquad \pi(x, t) = \rho(\xi, \tau)\mathrm{e}^{\mathrm{i}\omega t} \tag{10}$$

that leads to

$$\frac{\partial\rho(\xi, \tau)}{\partial\tau} = \frac{\hbar}{4m\omega}\frac{\partial^2\rho(\xi, \tau)}{\partial\xi^2} \tag{11}$$

which is a FP equation for a free particle. It is quite interesting to note that, if the real line used for fixing the position of the particle, the time axis and the very complex measure undergo a transformation corresponding to a harmonic oscillator, the particle executes a free motion in the new configuration. The situation in the case of displaced and forced oscillators is exactly the same. This apparently simple result acquires profound significance if we make note of the fact that the electromagnetic field is a linear superposition of a large number of harmonic oscillators over different frequencies.

3. Motion of an electron in a uniform magnetic field

In section 2 we have discussed the free electron motion from the QMTF point of view. We have shown that the derivation of the FP equation in QMTF leads to the Dirac equation in Weyl representation provided we use the transformation (7). While the factor in the right-hand side (rhs) of equation (7) may not appear to be significant, it is very crucial from the measure theoretic point of view. In fact, it identifies the ground state as the stationary state with all other states interpreted as quasi-stationary from a probabilistic view point, the connection to the observable positive-definite probability arising from the introduction of modulus measure (see Srinivasan (1997)). The next question we address is the *modus operandi* for the introduction of the electromagnetic field. The electromagnetic field is introduced in conventional treatment by the standard replacement

$$\boldsymbol{p} \to p - e\frac{\boldsymbol{A}}{c} \qquad \text{or} \qquad -\mathrm{i}\hbar\boldsymbol{\nabla} \to -\mathrm{i}\hbar\nabla - e\frac{\boldsymbol{A}}{c};$$

in fact this was the procedure adopted by Dirac. Unfortunately we cannot justify such a replacement in the FP system. Firstly, it has no probabilistic/measure theoretic basis; in fact it disturbs the delicate balance of the distribution of the complex/quaternion measure and leads to a different variational measure. Secondly, the possibility of interpreting the replacement as the addition of a connection coefficient to render the derivative as a covariant one is too much of an *ad hoc* assumption. It may be a reasonable idea to arrive at the same conclusion by an examination of the geometric properties of the measure structure. With this in mind, we attempt to model a specific situation wherein the electromagnetic field is of a simple nature such as, for example, a pure magnetic field; we simplify this still further by assuming that it is uniform. If B is the strength of the magnetic field which we assume to be in the z-direction, there is an extra drift in addition to that defined by equations (1)–(3). Thus we take the drift along the x- and y-directions respectively to be

$$-\mathrm{i}\frac{\omega}{2}(1+\lambda\sigma_3)(x+\mathrm{i}y) \qquad \text{and} \qquad -\frac{\omega}{2}(1+\lambda\sigma_3)(x+\mathrm{i}y),$$

where $\omega = eB/mc$. The drift coincides for $\lambda = 0$ with that used earlier (Srinivasan 2001) to deal with the case when spin is neglected. Thus instead of the set of equations (4) and (5) we now have

$$\begin{aligned}\partial_t\pi_+(\boldsymbol{x},t) = {}& -c\sigma\cdot\nabla\pi_+(\boldsymbol{x},t) + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y)\pi_+(\boldsymbol{x},t)\\ &+\left[\mathrm{i}\frac{mc^2}{\hbar}+\mathrm{i}\omega(1+\lambda\sigma_3)\right]\pi_+(\boldsymbol{x},t) - \mathrm{i}\frac{mc^2}{\hbar}\pi_-(\boldsymbol{x},t)\end{aligned} \tag{12}$$

$$\begin{aligned}\partial_t\pi_-(\boldsymbol{x},t) = {}& c\sigma\cdot\nabla\pi_-(\boldsymbol{x},t) + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y)\pi_-(\boldsymbol{x},t)\\ &+\left[\mathrm{i}\frac{mc^2}{\hbar}+\mathrm{i}\omega(1+\lambda\sigma_3)\right]\pi_-(\boldsymbol{x},t) - \mathrm{i}\frac{mc^2}{\hbar}\pi_+(\boldsymbol{x},t).\end{aligned} \tag{13}$$

Written in matrix form the above set of equations reads

$$\begin{aligned}\partial_t\begin{pmatrix}\pi_+(\boldsymbol{x},t)\\ \pi_-(\boldsymbol{x},t)\end{pmatrix} = {}& \Bigg\{-c\sigma\cdot\nabla\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \Bigg[\mathrm{i}\frac{mc^2}{\hbar}+\mathrm{i}\omega(1+\lambda\sigma_3)\\ &+\frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y)\Bigg]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \mathrm{i}\frac{mc^2}{\hbar}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Bigg\}\begin{pmatrix}\pi_+(\boldsymbol{x},t)\\ \pi_+(\boldsymbol{x},t)\end{pmatrix}.\end{aligned} \tag{14}$$

We now make a transformation which leads to a natural separation of large and small components:

$$\partial_t \begin{pmatrix} \phi_+(\boldsymbol{x},t) \\ \phi_-(\boldsymbol{x},t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pi_+(\boldsymbol{x},t) \\ \pi_-(\boldsymbol{x},t) \end{pmatrix}. \tag{15}$$

The above transformation also shows the connection between the present form and the standard Dirac form. Thus we have

$$\partial_t \begin{pmatrix} \phi_+(\boldsymbol{x},t) \\ \phi_-(\boldsymbol{x},t) \end{pmatrix} = \left\{ -c\sigma \cdot \nabla \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \left[\mathrm{i}\frac{mc^2}{\hbar} + \mathrm{i}\omega(1+\lambda\sigma_3) \right.\right.$$
$$\left.\left. + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y) \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mathrm{i}\frac{mc^2}{\hbar} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \phi_+(\boldsymbol{x},t) \\ \phi_+(\boldsymbol{x},t) \end{pmatrix}. \tag{16}$$

Expressed in components, we have

$$\partial_t\phi_+(\boldsymbol{x},t) = -c\sigma \cdot \nabla\phi_-(\boldsymbol{x},t) + \left[\mathrm{i}\omega(1+\lambda\sigma_3) + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y) \right]\phi_+(\boldsymbol{x},t) \tag{17}$$

$$\partial_t\phi_-(\boldsymbol{x},t) = -c\sigma \cdot \nabla\phi_+(\boldsymbol{x},t) + \left[\frac{2\mathrm{i}mc^2}{\hbar} + \mathrm{i}\omega(1+\lambda\sigma_3) \right.$$
$$\left. + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y) \right]\phi_-(\boldsymbol{x},t). \tag{18}$$

If in equation (18) we retain only the dominant terms, we have

$$\frac{2\mathrm{i}mc}{\hbar}\phi_-(\boldsymbol{x},t) = \sigma \cdot \nabla\phi_+(\boldsymbol{x},t) \tag{19}$$

and on substitution equation (17) becomes

$$\partial_t\phi_+(\boldsymbol{x},t) = \frac{\mathrm{i}\hbar}{2m}\nabla^2\phi_+(\boldsymbol{x},t) + \left[\mathrm{i}\omega(1+\lambda\sigma_3) + \frac{\omega}{2}(x+\mathrm{i}y)(1+\lambda\sigma_3)(\mathrm{i}\partial_x+\partial_y) \right]\phi_+(\boldsymbol{x},t). \tag{20}$$

At this stage it is worth noting that the above equation for the 'large component' can be derived directly by using the drift function together with the assumption that under the non-relativistic approximation there is a non-vanishing diffusion function which is a constant equal to $\mathrm{i}\hbar/2m$. To make contact with the usual Schrödinger–Pauli structure, we note that ϕ_+ is a function of σ_3 (quaternion variable) only. Thus we can make the transformation

$$\phi_+(\boldsymbol{x},t) = \psi(\boldsymbol{x},t)\exp\left[-\frac{m\omega}{4\hbar}(x^2+y^2)(1+\lambda\sigma_3) \right] \tag{21}$$

which in turn leads to

$$\mathrm{i}\hbar\partial_t\psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{x},t) + \left\{ -\frac{\omega\hbar}{2}(1+\lambda\sigma_3) - \frac{\hbar\omega}{2}L_z(1+\lambda\sigma_3) \right.$$
$$\left. + \frac{m\omega^2}{8}(1+\lambda\sigma_3)^2(x^2+y^2) \right\}\psi(\boldsymbol{x},t). \tag{22}$$

It is worth noting that the terms $-(\lambda\omega/2)\hbar\sigma_3(1+L_z)$ and $-(\hbar\omega/2)L_z$ together yield the correct magnetic moment interaction and magnetic energy for $\lambda = 1/2$. The ground-state energy is zero for $\sigma_3 = -1$ and $L_z = 1$ and there is a natural spin–orbit interaction built into the system. While equation (21) is a bridge between the conventional theory and QMTF, it is $\phi_+(\boldsymbol{x},t)$ that is of any significance from the QMT point of view.

Next we note that $\phi_+(\boldsymbol{x}, t)$ admits of a closed form solution. The quaternion analytic treatment of equation (20) leading to an explicit solution is given in the appendix. The solution in the cylindrical polar coordinate system is given by

$$\phi_+(\boldsymbol{x}, t) = \pi_0(z, t)\pi_1(r, \theta, t) \tag{23}$$

where $\pi_0(z, t)$ is the free particle propagation given by

$$\pi_0(z, t) = \left(\frac{m}{2\pi\hbar \mathrm{i}t}\right)^{1/2} \exp \mathrm{i}\frac{m(z - z_0)^2}{2\hbar t} \tag{24}$$

while $\pi_1(r, \theta, t)$ is given by

$$\pi_1(r, \theta, t) = \left(-\frac{m\mathrm{i}}{2\pi\hbar}\mathrm{e}^{\mathrm{i}\omega t(1+\lambda\sigma)}\tau^{-1}\right) \exp -\frac{m\omega}{2\hbar}AP(\rho, \theta, t) \tag{25}$$

$$\begin{aligned} P = \Bigg\{ &\left[\rho\cos\left(\frac{\omega t}{2}[1 + \lambda\sigma] + \theta\right) - r_0\cos\theta_0\right]^2 \\ &+ \left[\rho\sin\left(\frac{\omega t}{2}[1 + \lambda\sigma] + \theta\right) - r_0\sin\theta_0\right]^2\Bigg\}\mathrm{e}^{-\mathrm{i}\omega t(1+\lambda\sigma)} \end{aligned} \tag{26}$$

$$A = \frac{(1 + \lambda\sigma)(1 - \mathrm{e}^{-\mathrm{i}\omega t}\cos\lambda\cos\omega t) - \mathrm{i}(m\sigma)\mathrm{e}^{-\mathrm{i}\omega t}\sin\lambda\omega t}{1 + \mathrm{e}^{-2\mathrm{i}\omega t} - 2\mathrm{e}^{-\mathrm{i}\omega t}\cos\lambda\omega t} \tag{27}$$

$$\rho = r\mathrm{e}^{\mathrm{i}(\omega t/2)(1+\lambda\sigma)} \tag{28}$$

where σ is used in the place of σ_3 for notational simplicity.

It is interesting to note that $\pi_1(r, \theta, t)$ has a limit as $t \to \infty$ under the assumption $(\omega \to \omega - \mathrm{i}\epsilon, \epsilon > 0)$ and is determined by

$$\lim AP = (1 + \lambda\sigma)\{[x - \tfrac{1}{2}(x_0 - \mathrm{i}y_0)]^2 + [y - \tfrac{\mathrm{i}}{2}(x_0 - \mathrm{i}y_0)]^2\}. \tag{29}$$

The normalization factor in equation (25) also has the correct limit to make the total measure unity. As stated earlier, the choice $\lambda = 1/2$ leads to the correct value of the magnetic moment of the electron; it also ensures correct asymptotic behaviour to ensure convergence of the measure density. The solution given by equations (23)–(25) can be compared with that obtained by the use of Schrödinger theory. The motion of an electron in a uniform magnetic field was dealt with by Landau (1930) and Johnson and Lippmann (1946) who had obtained explicitly the eigenfunctions of the canonical momentum operators; the parameters x_0, y_0 used in equation (29) correspond to the guiding centre coordinates. By using the Fourier transform technique, it can be easily shown that there is a correlation between the momentum in the x-direction (y-direction) and $y_0(x_0)$. That such an entanglement is possible has been noticed only very recently by Fan *et al* (2000) who have provided a demonstration by using the standard operator theory in Fock space. However, the result relates to the special case when spin motion is neglected. The solution as given by equation (29) brings out this aspect in an equally transparent manner in a more general setting when spin motion is included.

We next come to the main theme of the paper. The solution given above is arrived at in the appendix by the transformation of the independent variables from (r, θ) to (ρ, ϕ) specified by equation (A.3) and $t \to \tau$ given by equation (A.10) along with a transformation of the function π_1 itself given by equation (A.9). The cumulative effect of such a transformation is to yield the FP equation (A.11) for a free particle (with no drift). In other words, in a new frame of reference given by equations (A.3) and (A.10), the quaternion measure density as transformed by equation (A.9) corresponds to that of a free particle subject to pure diffusion. Thus the magnetic field is a manifestation of the geometry of the product space of space–time and the quaternion space over which the measure is defined. Since this is an important conclusion, we explore whether the same conclusion can be arrived at in a more general situation when the motion is relativistic as described by equation (14). We rewrite equation (14) using the cylindrical polar coordinate system:

$$\partial_t \begin{pmatrix} \pi_+(z,r,\theta,t) \\ \pi_-(z,r,\theta,t) \end{pmatrix} = \left\{ -c\sigma\cdot\nabla \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \left[\frac{\mathrm{i}mc^2}{\hbar} + \mathrm{i}\omega(1+\lambda\sigma_3) \right.\right.$$
$$\left.\left. + \frac{\omega}{2}(1+\lambda\sigma_3)(\mathrm{i}r\partial_r + \partial_\theta) \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\mathrm{i}mc^2}{\hbar} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \pi_+(z,r,\theta,t) \\ \pi_-(z,r,\theta,t) \end{pmatrix}. \tag{30}$$

We next transform to characteristic coordinates

$$r\mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)} = \xi, \qquad \frac{\omega t}{2}(1+\lambda\sigma_3) - \theta = \phi. \tag{31}$$

Using left multiplication for the chain rule of the derivative, we find that equation (30) reduces to

$$\partial_t \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = \left\{ -c\hat{\zeta}\cdot\nabla \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \left[\frac{imc^2}{\hbar} + \mathrm{i}\omega(1+\lambda\sigma_3) \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right.$$
$$\left. - \frac{\mathrm{i}mc^2}{\hbar} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} \tag{32}$$

where

$$\hat{\zeta} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3), \qquad \hat{\zeta}_1 = \sigma_1 \mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)}, \qquad \hat{\zeta}_2 = \sigma_2 \mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)}, \qquad \hat{\zeta}_3 = \sigma_3. \tag{33}$$

If we now set

$$\pi_\pm = \mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)+(\mathrm{i}mc^2/\hbar)t}\psi_\pm, \tag{34}$$

we obtain

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = -\left\{ c\zeta\cdot\nabla \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\mathrm{i}mc^2}{\hbar} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{35}$$

where

$$\begin{aligned} &\zeta = (\zeta_1, \zeta_2, \zeta_3), \qquad \zeta_1 = \mathrm{e}^{-\mathrm{i}\omega t\lambda\sigma_3}\sigma_1 \mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)+\mathrm{i}\omega t\lambda\sigma_3}, \\ &\zeta_2 = \mathrm{e}^{-\mathrm{i}\omega t\lambda\sigma_3}\sigma_2 \mathrm{e}^{(\mathrm{i}\omega t/2)(1+\lambda\sigma_3)+\mathrm{i}\omega t\lambda\sigma_3}, \qquad \zeta_3 = \sigma_3. \end{aligned} \tag{36}$$

Equation (35) represents the free particle FP equation in QMTF in a new frame of reference where the quaternions have undergone a transformation given by equation (36) with the coordinates having been transformed according to equation (31). Thus we can conclude that the magnetic force is a manifestation of the geometrical transformation of the product space of space–time and the quaternion space. While the scenario may appear to be similar to that in the theory of gauge transformation, there are fundamental differences. In modern gauge theory, forces of electromagnetic type (Abelian gauge) and other types (cf electroweak) are assumed to arise from the connection coefficients of an associated charge space; the additional forces that arise due to geometric transformation have no observable physical effect due to gauge principle. However, in QMTF the situation is different; the transformation is on the product space of space–time and quaternion space whereas the standard gauge transformation is on the product space of space–time and a hypothetical charge space. Moreover, in the standard approach, the probabilistic inferences are made on the basis of Born interpretation which is externally imposed; on the other hand, in QMTF, probabilistic inference or rather Born interpretation automatically flows. Thus, in QMTF the basic forces of electromagnetic type are themselves manifestations of the geometric transformation of the product space. Of course we have dealt with special cases and have shown that there are geometric transformations that lead to a frame of reference wherein the particle has a free motion. For instance, in the case of a more general magnetic field (non-uniform), further research is necessary to arrive at the appropriate FP equation. While on an intuitive basis it may be plausible, only with further research in this direction can an affirmative statement in the most general form be made and, until then, our statement has the status of a conjecture.

4. Summary and conclusions

In this paper we have reviewed several aspects of the extended measure theory approach with special reference to some facets of quantum phenomena. The very complex nature of the measure structure comes in handy to explain interference and violation of Bell's inequalities. On the other hand, the zitterbewegung character is appropriately modelled in terms of a two-valued quaternion measure space, the quaternion nature of the measure being able to describe aptly the intrinsic spin characteristics. The description of the harmonic oscillator in terms of the complex measure is shown to have intrinsic geometric properties. In particular it is shown that there is a transformation of the product space of space–time and complex plane describing the measure in which the particle executes free diffusion. Since the electromagnetic field can be considered as a weighted sum of independent oscillators distributed over different modes, it is reasonable to conjecture the field itself as a manifestation of the geometric properties of the product space. Encouraged by this, we have examined the motion of a spinning particle under the influence of a uniform magnetic field. The FP equation for the quaternion measure density is derived using appropriate quaternion drifts; the non-relativistic approximation is shown to lead to the conventional Schrödinger–Pauli structure with the spin–orbit interaction leading to the correct value for the magnetic moment of the particle. A closed form solution for the quaternion measure density is provided and as a bi-product we arrive at the appropriate geometric transformation of the product space leading to the preferred frame of reference wherein the particle executes free diffusion. Finally, it is shown that, when the motion is relativistic, there is a frame of reference and a base space of quaternions over which the particle disperses freely. It is to be noted that this characteristic is distinct from the familiar one in the modern gauge theory; the difference lies in the fact that forces of electromagnetic origin are shown to arise by geometrical transformation. It is to be noted that, in conventional modern gauge theory, a hypothetical charge space is to be invoked for identification of connection coefficients with electromagnetic/electroweak forces. On the other hand, in QMTF, we do not have to go beyond the basic measure structure, the product space of space–time and quaternion space having enough degrees of freedom to generate forces. In the specific instance of a uniform magnetic field, we have shown that it is indeed possible to arrive at a frame of reference in which the magnetic field disappears. In other words, the magnetic field is shown to be a manifestation of the geometric properties of the product space. Since we have not considered the most general situation, our statement must be taken to be a conjecture.

Appendix

Our object is to obtain a closed form solution for $\phi_+(x, t)$ satisfying equation (20). At the outset, we note that ϕ_+ is a function of σ_3 only and hence in the analysis there are no problems of non-commutativity. We use cylindrical polar coordinates and the functional symbol $\pi(z, r, \theta, t)$;

then equation (20) becomes

$$\partial_t \pi(z, r, \theta, t) = \left\{ \mathrm{i}\omega(1 + \lambda\sigma) + \frac{\omega}{2}(1 + \lambda\sigma)(\mathrm{i}r\partial_r + \partial_\theta) + \frac{\mathrm{i}\hbar}{2m}\nabla^2 \right\} \pi(z, r, \theta, t) \tag{A.1}$$

where for convenience we use σ in place of σ_3. We introduce the characteristics by

$$-\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}r}{\mathrm{i}(\omega/2)(1 + \lambda\sigma)r} = \frac{\mathrm{d}\theta}{(\omega/2)(1 + \lambda\sigma)} \tag{A.2}$$

whose integrals are given by

$$r \exp \mathrm{i}\frac{\omega t}{2}(1 + \lambda\sigma) = \rho, \qquad \frac{\omega t}{2}(1 + \lambda\sigma) = \phi. \tag{A.3}$$

Using (ρ, ϕ) in place of (r, θ), we find that equation (A.1) reduces to

$$\partial_t \pi = \frac{\mathrm{i}\hbar}{2m}\{\partial_{zz} + \mathrm{e}^{\mathrm{i}\omega t(1+\lambda\sigma)}\nabla^2 + \mathrm{i}\omega(1 + \lambda\sigma)\}\pi \tag{A.4}$$

where ∇^2 is the two-dimensional Laplacian with respect to the polar coordinates (ρ, ϕ). We solve equation (A.4) by separation of variables and set

$$\pi = \pi_0(z, t)\pi_1(\rho, \phi, t) \tag{A.5}$$

where π_0 is scalar valued and π_1 is a function of the quaternion σ. Setting

$$\partial_t \pi_0 - \frac{\mathrm{i}\hbar}{2m}\partial_{zz}\pi_0 = f(t)\pi_0 \tag{A.6}$$

$$\partial_t \pi_1(\rho, \phi, t) = \left[\mathrm{i}\omega(1 + \lambda\sigma) + \frac{\mathrm{i}\hbar}{2m}\mathrm{e}^{\mathrm{i}\omega t(1+\lambda\sigma)}\nabla^2 - f(t) \right] \pi_1(\rho, \phi, t). \tag{A.7}$$

Thus the arbitrary function $f(t)$ becomes eliminated and $\pi_0(z, t)$ corresponds to the free particle kernel in the variable z:

$$\pi_0(z, t) = \left(\frac{m}{2\pi\hbar \mathrm{i}t} \right)^{1/2} \exp + \frac{\mathrm{i}m(z - z_0)^2}{2\hbar t}. \tag{A.8}$$

Setting

$$\pi_1(\rho, \phi, t) = \chi(\rho, \phi, t) \exp[\mathrm{i}\omega t(1 + \lambda\sigma)] \tag{A.9}$$

$$\tau = \int_0^t \mathrm{e}^{\mathrm{i}\omega u(1+\lambda\sigma)}\mathrm{d}u \tag{A.10}$$

we find

$$\partial_\tau \chi = \frac{\mathrm{i}\hbar}{2m}\nabla^2\chi \tag{A.11}$$

which is the FP equation for a free particle. Thus the solution for π_1 can be written as

$$\pi_1(\rho, \phi, t) = \left[\left(\frac{-m\mathrm{i}}{2\pi\hbar} \right) \mathrm{e}^{\mathrm{i}\omega t(1+\lambda\sigma)}\tau^{-1} \right] \exp\left\{ -\frac{m\omega}{2\hbar}A[(\rho\cos\theta - \rho_0\cos\theta_0)^2 + (\rho\sin\theta - \rho_0\sin\theta_0)^2]\mathrm{e}^{-\mathrm{i}\omega t(1+\lambda\sigma)} \right\} \tag{A.12}$$

where

$$A = \frac{(1 + \lambda\sigma)(1 - \mathrm{e}^{-\mathrm{i}\omega t}) - \mathrm{i}(\lambda + \sigma_1)\mathrm{e}^{-\mathrm{i}\omega t}\sin\lambda\omega t}{1 + \mathrm{e}^{-2\mathrm{i}\omega t} - 2\mathrm{e}^{-\mathrm{i}\omega t}\cos\lambda\omega t}. \tag{A.13}$$

We now simplify the term in square brackets under exponential and finally obtain

$$\pi_1(\rho,\phi,t) = \left(-\frac{m\mathrm{i}}{\pi\hbar}\mathrm{e}^{\mathrm{i}\omega t(1+\lambda\sigma)}\tau^{-1}\right)\exp\left\{-\frac{m\omega}{2\hbar}AP(\rho,\phi,t)\right\} \tag{A.14}$$

$$\begin{aligned} P(\rho,\phi,t) = \Bigg\{ & \left(\rho\cos\left(\frac{\omega t}{2}[1+\lambda\sigma]+\theta\right) - r_0\cos\theta_0\right)^2 \\ & + \left(\rho\sin\left(\frac{\omega t}{2}[1+\lambda\sigma]+\theta\right) - r_0\sin\theta_0\right)^2\Bigg\}\mathrm{e}^{-\mathrm{i}\omega(1+\lambda\sigma)}. \end{aligned} \tag{A.15}$$

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